# §1.3 Back to the Path integral – disturbing the vacuum

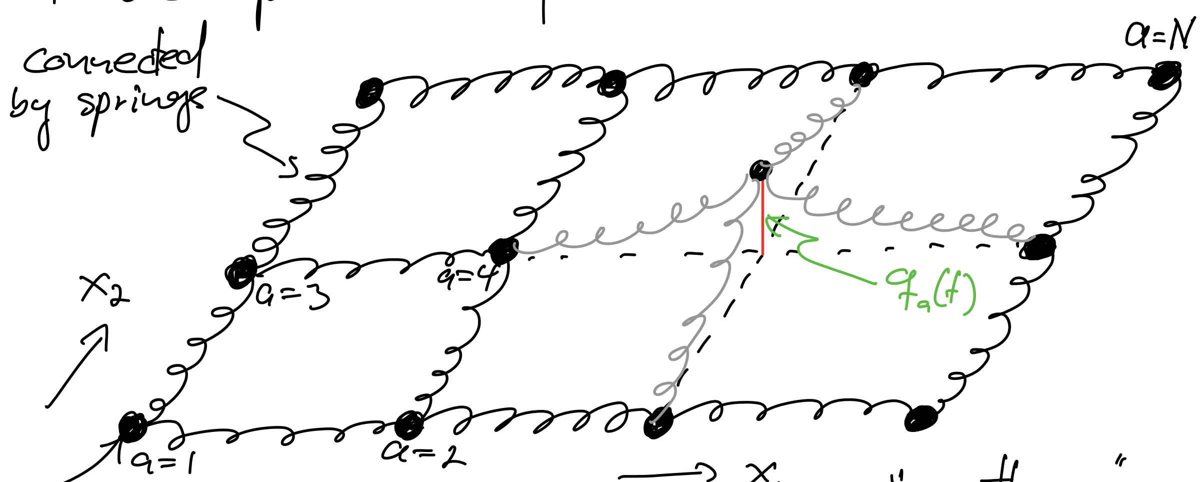
Last time:
continuum limit of system of N particles

$q_a(t)$, $a = 1, \dots, N$

generalized coordinate

$\xrightarrow{N\to\infty}$ field theory with field
$\varphi(t, \vec{x})$, $\vec{x} \in \mathbb{R}^{D-1}$

When $D-1 = 2$ (or 3) there is a nice intuitive picture for this:




point masses $m$
with position label $a$
and vertical displacement $q_a(t) \xrightarrow{N\to\infty} \varphi(t, \vec{x})$

This picture is very suitable for describing the vacuum!

$\longrightarrow$ state where all point masses are on the plane, i.e. $q_a = 0 \ \forall a$

Disturbing the vacuum:

pushing on the mass labeled by a

$\longrightarrow$ add term $J_a(t)\, q_a$ to potential

More generally:

$$\text{add } \sum_a J_a(t)\, q_a \xrightarrow{N \to \infty} \underbrace{J(x)}_{\text{"source function" } J(t,\vec{x})} \varphi(x)$$

$\longrightarrow$ creates wave packets going off here and there

modified path integral (take $D=4$):

$$Z = \int \mathcal{D}\varphi\, e^{i\int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + J(x)\varphi(x)\right]}$$

Let's do free field theory:

$$\mathcal{L}(\varphi) = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right]$$ "Gaussian theory"

$\longrightarrow$ equation of motion: $(\partial^2 + m^2)\varphi = 0$
"Klein-Gordon" eq.

solution: $\varphi(\vec{x}, t) = e^{i(\omega t - \vec{k}\cdot\vec{x})}$
with $\omega = \vec{k}^2 + m^2 \quad (\hbar = 1)$
"energy-momentum relation"

Using path integral, we compute

$$Z[J] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi\right)}$$

$$= \int \mathcal{D}\varphi \, e^{i\int d^4x \left[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi\right]} \quad (*)$$

recall:

$$\int d^N q \; e^{\left(\frac{i}{2}\right) q^T A q + i J\cdot q}$$

$q = (q_1, \dots, q_N)$ N-vector

$$= \left(\frac{(2\pi i)^N}{\det A}\right)^{\frac{1}{2}} e^{-\left(\frac{i}{2}\right) J^T A^{-1} J} \quad \text{(homework)}$$

role of $A$ in $(*)$ is $-(\partial^2 + m^2)$

"inverse" is computed by solving

$$-(\partial^2 + m^2)\, D(x-y) = \delta^{(4)}(x-y) \qquad (**)$$

$$\rightarrow Z[J] = \underset{\text{const.}}{C}\, e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y)}$$

$$=: C\, e^{iW[J]}$$

Setting $C = Z[J=0]$ gives

$$Z[J] = Z[0]\, e^{iW[J]}$$

For the integral in $(*)$ to converge, let $m^2 \longmapsto m^2 - i\varepsilon$

$\rightarrow$ obtain term $e^{-\varepsilon\int d^4x\, \varphi^2}$, $\varepsilon \ll 1$

Let us solve $(**)$ by going to momentum space:

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}$$

$$\rightarrow \text{solution}: \; D(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

To evaluate $D(x)$, we first integrate over $k^0$ by contour integration:

$$D(x) = (2\pi)^{-4} \int d^4k \, \frac{\exp(i k \cdot x)}{k^2 + m^2 - i\varepsilon}$$

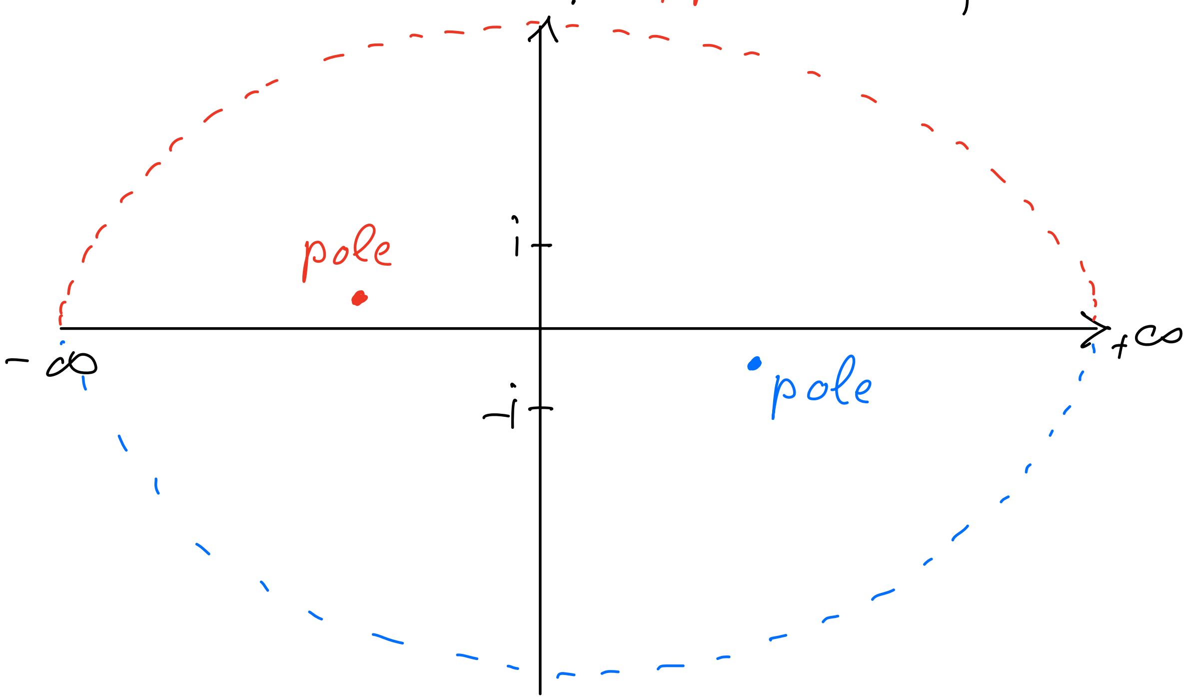
$$= (2\pi)^{-4} \int d^3k \int dk_0 \, \frac{\exp\left(i(k_0 x_0 - \vec{k} \cdot \vec{x})\right)}{k_0^2 - \vec{k}^2 - m^2 + i\varepsilon}$$

$\longrightarrow$ has poles at

$$k_0 = \pm \sqrt{\underbrace{\vec{k}^2 + m^2}_{= \omega_k^2} - i\varepsilon} \quad , \quad \omega_k := \sqrt{\vec{k}^2 + m^2}$$

$$= \pm \omega_k \mp \frac{i\varepsilon}{2\omega_k} + \mathcal{O}(\varepsilon^2)$$

$\longrightarrow$ is in lower/upper half plane




Close contour in lower/upper half-plane for $x_0 \longrightarrow -\infty$ / $x_0 \longrightarrow +\infty$

Thus

$$D(x) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big[ e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \Theta(x^0) + e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \Theta(-x^0) \Big]$$

$D(x)$ is known as the "propagator"

$\longrightarrow$ it describes the amplitude for a disturbance in the field to travel from origin to $x$.

<u>Causality:</u>

One can show:

- $x = (t, 0)$, $t > 0$ (within light-cone)

  $D(x) \underset{t\to\infty}{\sim} e^{-imt}$ "oscillatory"

- $x = (0, \vec{x})$ (outside light-cone)

  $D(x) \underset{r\to\infty}{\sim} e^{-mr}$

  amplitude can "leak" outside of light-cone over distance $r \sim \frac{1}{m}$

# §1.4 From Field to Particle to Force

## i) From field to particle

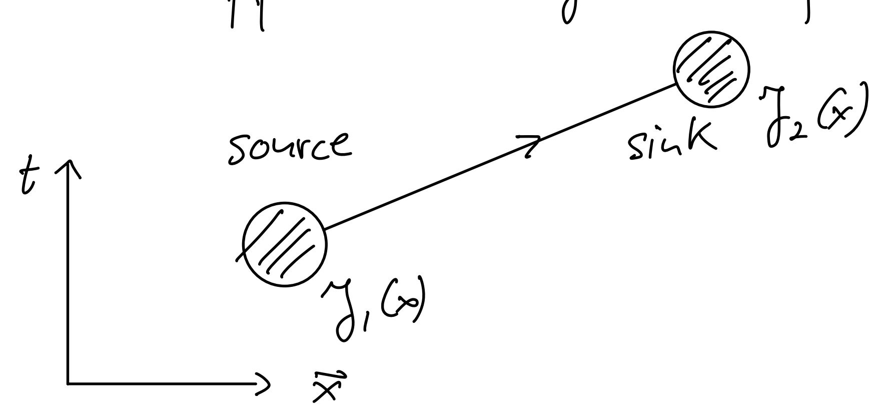
Recall
$$W[J] = -\frac{1}{2}\int d^4x \int d^4y \; J(x)\, D(x-y)\, J(y)$$

Use Fourier transform
$$J(k) := \int d^4x \; e^{-ik\cdot x} J(x)$$

$$\longrightarrow \quad W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \, J(k)^* \frac{1}{k^2-m^2+i\varepsilon} J(k)$$

(note: $J(k)^* = J(-k)$ for real $J(x)$)

Consider now $J(x) = J_1(x) + J_2(x)$ with $J_1$ and $J_2$ localized in two different regions of spacetime:



Let us focus on the term

$$W[J] = \dots - \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J_2^*(k) \frac{1}{k^2 - m^2 + i\varepsilon} J_1(k) \dots$$

→ significant contribution iff

- $J_1(k)$ and $J_2(k)$ overlap in a large region
- $k^2 - m^2 \sim 0$ in that region

→ energy-momentum relation of particle of mass $m$ traveling from region 1 to region 2

2) From particle to force

Imagine now $J(x) = J_1(x) + J_2(x)$

with $J_a(x) = \delta^{(3)}(x - x_a)$

(two spikes localized at $\vec{x}_1$ and $\vec{x}_2$, time-indep. )

$$\rightarrow W[J] = - \int dx^0 dy^0 \int \frac{dk^0}{2\pi} e^{ik^0(x^0 - y^0)}$$

we neglect the self-interactions $J_1^2$ and $J_2^2$ here

$$\times \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\varepsilon}$$

$$= \left(\int dx^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

recall $Z = C e^{iW[J]}$

$$= \langle 0| e^{-iHT} |0\rangle_J$$

$$= e^{-iE_J T}$$

$\longrightarrow \; iW[J] = iE_J T$

identify $\left(\int dx^0\right)$ with $T$ and

$$E_J = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

$\rightarrow$ ground state energy has been lowered!

$\longrightarrow$ attractive force between the two sources!

$r = |\vec{x}_1 - \vec{x}_2|$

We now compute

$$E_J = -\frac{1}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{k^2 + m^2} \underbrace{\int_0^\pi d\theta \sin\theta \, e^{ikr\cos\theta}}_{= (e^{ikr} - e^{-ikr})/ikr}$$

$$= \frac{i}{(2\pi)^2} \int_{-\infty}^{\infty} dk \, \frac{k}{k^2+m^2} \, \frac{e^{ikr}}{r}$$

View as complex contour integral

$\longrightarrow$ has poles at $k = \pm im$

close contour in upper half-plane

$\longrightarrow$ pick residue at $k = im$:

$$E_g = -\frac{1}{4\pi r} e^{-mr}$$